Journal of Algebra 347 (2011) 173–183

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Arcs schemes, derivations and Lipman's theorem

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ARTICLE INFO

Article history:
Received 28 May 2011
Available online 15 September 2011
Communicated by Steven Dale Cutkosky

Keywords:
Differential algebra
Algebraic geometry
Derivation module on varieties
Arcs scheme

ABSTRACT

In this article, we prove that a projective plane curve defined over a subfield k of $\mathbf{C}$ is smooth over k if and only if its associated arcs scheme is reduced.

1. Introduction

If X is an algebraic variety over a field, the *arcs scheme* (resp. the *jets scheme* of level n) associated with X is the scheme over k, in general not of finite type, (resp. the variety over k) that can be realized as the space of the formal solutions (resp. the polynomial solutions with degree at most n) in one variable of the local equations of X.

In [4], Mircea Mustaţă has proved a remarkable theorem that relates the singularities of the variety X with the irreductibility of its jets schemes.

Following this idea, we are interesting in the following original question:

Question 1. Does there exist an interpretation in terms of the geometry of the variety X of the fact that its associated arcs scheme is reduced?

Up to now, we know only few things about that. It is easy to verify that, if the arc space $\mathcal{L}(X)$ is reduced, then the algebraic variety X also is reduced; and it is easy to find examples of reduced varieties with a non-reduced associated arcs scheme (see Section 4.8).

In the present article, we show the following precise statement, that clarifies the situation for plane curves (see Theorem 4).

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.08.018

Theorem 1. *Let k be a subfield of $\mathbf{C}$. Let X be a plane curve over k. Then the curve X is smooth over k if and only if its arcs scheme $\mathcal{L}(X)$ is reduced.*

To prove this statement, we introduce a general method, which is totally new (see Section 4.6). That method allows to exhibit, from any given derivation on X, an explicit nilpotent element of the ring of regular functions of $\mathcal{L}(X)$. This element is in particular non-trivial, for a "good" choice of the derivation, and if X is reduced and non-singular (see Theorem 3). By this way, the proof also uses Lipman's theorem on derivation modules on varieties (see [3] and Theorem 2). In the last Section 4.8, we present some complete examples to illustrate our method.

The present article is self-contained. In Section 3, we recall results on arcs schemes and derivations, centered around our question.

2. Notations, conventions

In this article, k is a field of characteristic zero, with algebraic closure k^{alg}. We denote by $k[\![X_1, \ldots, X_N]\!]$ (resp. $\mathbf{C}\{x, y\}$) the ring of formal power series in the variables X_i with coefficients in k (resp. the ring of convergent power series at the origin of $\mathbf{C}^2$).

An (algebraic) *variety* is a separated scheme of finite type over k. A *curve* over k is a variety over k purely of dimension one. A *plane curve* (resp. *affine plane curve*) over k is a hypersurface of $\mathbf{P}^2_k$ (resp. $\mathbf{A}^2_k$). In particular, an affine plane curve is uniquely determined, up to multiplication by some $\lambda \in k \backslash \{0\}$, by the datum of a polynomial $F \in k[x, y] \backslash k$. An affine plane curve is *reduced* if and only if its presentation F is *reduced*, i.e., has only simple irreducible factors.

If X is a variety, we denote by $\mathcal{L}(X)$ the arcs scheme associated to X. If $X = \mathrm{Spec}(A)$, we set $A_\infty := \mathcal{O}_{\mathcal{L}(X)}(\mathcal{L}(X))$. An element of A_∞ is called a *differential polynomial*.

If A is a k-algebra, a *derivation* over k of A is a k-linear map $D : A \to A$ that verifies Leibniz rule, i.e.,

$$D(ab) = aD(b) + bD(a). \tag{1}$$

We denote by $\mathrm{Der}_k(A) := \mathrm{Der}_k(A, A)$ the set of derivations of A over k. If $F \in k[x, y]$, we denote by δ_F the derivation over k of A defined by $\partial_y(F)\partial_x - \partial_x(F)\partial_y$. Note that $\delta_F(F) = 0$. A *reduced* derivation $D \in \mathrm{Der}_k(k[x, y])$ is a derivation $D = a\partial_x + b\partial_y$, such that $a, b \in k[x, y]$ have no common factor. A k-algebra endowed with a derivation over k is a *differential k-algebra*. If (A, D) is a differential k-algebra, and if $S \subset A$, we denote by $[S]$ the *differential ideal* generated by S. It is the smallest ideal of A, containing S, such that $D(s) \in [S]$, for any $s \in [S]$. Its radical is denoted by $\{S\}$. Since the characteristic of k is 0, $\{S\}$ is again a differential ideal of A.

3. An overview on arcs spaces and derivations

In this section, we recall some important facts on arcs schemes and derivations.

3.1. The arcs scheme associated with an algebraic variety

We present here some useful classical properties (see [5] or [12] for example). Let k be a field. Let Y be a scheme over k. We denote by $\widehat{Y} := \widehat{Y \times_k k[\![T]\!]}$ the T-adic completion of the $k[\![T]\!]$-scheme $Y \times_k k[\![T]\!]$.

Definition 1. Let k be a field. Let X be a variety over k. There exists a unique separated k-scheme $\mathcal{L}(X)$, defined up to a unique isomorphism of k-schemes, such that, for any k-scheme Y, there exists a natural bijection:

$$\mathrm{Hom}_k\big(Y, \mathcal{L}(X)\big) \to \mathrm{Hom}_{\mathrm{Spf}(k[\![t]\!])}(\widehat{Y}, \widehat{X}).$$

This k-scheme $\mathcal{L}(X)$ is called the *arcs scheme* of X.

The k-scheme $\mathcal{L}(X)$ is endowed with a morphism of k-schemes $\pi_0 : \mathcal{L}(X) \to X$, called *projection*, that admits a *section* $s_0 : X \to \mathcal{L}(X)$. In the same way, we define the *jets scheme of level n of X*, for any $n \in \mathbf{N}$, to be the unique k-variety $\mathcal{L}_n(X)$ verifying

$$\mathrm{Hom}_k\big(Y, \mathcal{L}_n(X)\big) \cong \mathrm{Hom}_{\mathrm{Spec}(k)}\big(Y \times_k \mathrm{Spec}\big(k[t]/\big(t^{n+1}\big)\big), X\big).$$

In particular, note that X is canonically isomorphic to $\mathcal{L}_0(X)$.

Proposition 3.2. *Let k be a field. Let X be a variety over k.*

a) *For any field extension F of k, we have* $\mathcal{L}(X)(F) \cong X(F[\![T]\!])$.
b) *Let* $f : Y \to X$ *be an open immersion. Then the induced morphism of k-schemes* $\mathcal{L}(f) : \mathcal{L}(Y) \to \mathcal{L}(X)$ *is an open immersion.*
c) *If X is smooth over k, then* $\mathcal{L}(X)$ *is reduced.*

When $X = \mathrm{Spec}(k[X_1, \ldots, X_N]/I)$, we can give two concrete equivalent descriptions of $\mathcal{L}(X)$.

3.2.1. The description coming from the definition

Let A be any k-algebra. Let us fix $(f_1, \ldots, f_m)$ a presentation of the ideal I. Let

$$\varphi(T) = \left(\sum_{j\geqslant 0}\varphi_{1,j}T^j, \ldots, \sum_{j\geqslant 0}\varphi_{N,j}T^j\right) \in \big(A[\![T]\!]\big)^N.$$

For any integer s, $1 \leqslant s \leqslant m$, we have, in $A[\![T]\!]$,

$$f_s\big(\varphi(T)\big) = \sum_{\nu\geqslant 0} F_{s,\nu}\big((\varphi_{i,j})_{1\leqslant i\leqslant N, 0\leqslant j\leqslant \nu}\big)T^\nu.$$

It is then easy to verify the following statement.

Lemma 3.3. *Let k be a field and let* $I := (f_1, \ldots, f_m)$ *be an ideal of* $k[X_1, \ldots, X_N]$. *If* $X = \mathrm{Spec}(k[X_1, \ldots, X_N]/I)$, *then*

$$\mathcal{L}(X) \cong \mathrm{Spec}\big(k\big[(X_{1,j})_{j\in\mathbf{N}}, \ldots, (X_{N,j})_{j\in\mathbf{N}}\big]\big/\big(F_{s,\nu}\big((X_{i,j})_{i,j}\big)\big)_{1\leqslant s\leqslant m, \nu\geqslant 0}\big).$$

3.3.1. The description coming from differential algebra

For this second presentation, we adopt Ritt–Kolchin's point of view. Let $k\{X_1, \ldots, X_N\}$ be the differential k-algebra defined as the k-algebra

$$k\big[(X_{1,j})_{j\in\mathbf{N}}, \ldots, (X_{N,j})_{j\in\mathbf{N}}\big]$$

endowed with the derivation Δ over k defined by $\Delta(X_{i,j}) = X_{i,j+1}$, for any integer i, $1 \leqslant i \leqslant N$, and any integer j, $j \in \mathbf{N}$. It is then easy to verify the following statement.

Lemma 3.4. *Let k be a field of characteristic zero and let I be an ideal of* $k[X_1, \ldots, X_N]$. *If* $X = \mathrm{Spec}(k[X_1, \ldots, X_N]/I)$, *then*

$$\mathcal{L}(X) \cong \mathrm{Spec}\big(k\{X_1, \ldots, X_N\}/[I]\big).$$

Example 1. Let k be a field of characteristic zero. Let X be the hypersurface of $\mathrm{Spec}(k[x,y])$ defined by the datum of xy. Then we see that, from the first point of view, the associated arcs scheme is defined by the infinite system (S_1) represented by $X_0Y_0=0$, $X_1Y_0+X_0Y_1=0$, $X_2Y_0+X_1Y_1+X_0Y_2=0,\ldots.$ On the other hand, the second point of view gives the following system $X_0Y_0=0$, $X_1Y_0+X_0Y_1=0$, $X_2Y_0+2X_1Y_1+X_0Y_2=0,\ldots$ that we denote by (S_2). Note that the system (S_1) is equivalent to (S_2) by the change of variables $*_i\mapsto *_i/i!$.

3.5. The module of derivations on a ring

Let $X=\mathrm{Spec}(A)$ be an affine variety. A key ingredient of Section 4.6 and of our method is the use of derivations over k of A. The set $\mathrm{Der}_k(A)$ can be endowed with a structure of A-module, in the usual way.

Lemma 3.6. *Let G be a (finite) set of generators for the k-algebra A and let $D\in\mathrm{Der}_k(A)$. Then D is completely determined by the datum of the elements $D(g)$, $g\in G$.*

Example 2. Let $X=\mathbf{A}_k^n$ and $A=k[X_1,\ldots,X_n]$. Then we have

$$\mathrm{Der}_k(A)\cong A\partial_{X_1}\oplus\cdots\oplus A\partial_{X_n}.$$

Let A be a k-algebra of finite type. The A-module $\mathrm{Der}_k(A)$ can be linked to the regularity of A. Indeed, it also can be defined as the dual module $\mathrm{Hom}_A(\Omega^1_{A/k},A)$ of the Kälher differentials of A. Let $\mathfrak{p}$ be a prime ideal of A. Since $A_\mathfrak{p}$ is flat over A and $\Omega^1_{A/k}$ of finite type, we can identify $(\mathrm{Der}_k(A))_\mathfrak{p}$ and $\mathrm{Der}_k(A_\mathfrak{p})$. Let $x\in X=\mathrm{Spec}(A)$ be the point corresponding to the prime ideal $\mathfrak{p}$ of A. One knows that if X is smooth over k at x (or equivalently $A_\mathfrak{p}$ is regular, since k is perfect), then $\Omega^1_{A_\mathfrak{p}/k}=\Omega^1_{A/k}\otimes_A A_\mathfrak{p}$ is a free A_p-module, and so $\mathrm{Der}_k(A_\mathfrak{p})$ is free too.

Conjecture 1 *(Zariski–Lipman). Let X be a reduced variety over k. Let $x\in X$. The variety X is smooth over k at x if and only if $\mathrm{Der}_k(\mathcal{O}_{X,x})$ is a free $\mathcal{O}_{X,x}$-module.*

Up to knowledge, this conjecture is still open. (See [3] for some particular cases where it holds true.) In this direction, Joseph Lipman proves the following theorem (see [3, Theorem 1]).

Theorem 2 *(Lipman). Let k be a field of characteristic zero. Let X be a reduced variety over k and $x\in X$. If $\mathrm{Der}_k(\mathcal{O}_{X,x})$ is a free $\mathcal{O}_{X,x}$-module, then x is a normal point of X.*

In the proof of Theorem 4, we will use, in a crucial way, the following direct consequence of Theorem 2.

Remark 1. a) Zariski–Lipman's conjecture holds true for curves over k.

b) Let $X=\mathrm{Spec}(A)$ be a reduced affine variety. If there exists $\delta\in\mathrm{Der}_k(A)$ such that $\mathrm{Der}_k(A)=A\delta$, then X is a normal variety. Indeed, let $x\in X$ be a point of X corresponding to the prime ideal $\mathfrak{p}$ of A. We have to prove that $A_\mathfrak{p}$ is normal (i.e., integrally closed in its total quotient ring). But, by the assumption, $\mathrm{Der}_k(A_\mathfrak{p})\cong\mathrm{Der}_k(A)\otimes_A A_\mathfrak{p}\cong A_\mathfrak{p}\delta_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. (Here $\delta_\mathfrak{p}$ denotes the unique extension of δ to $A_\mathfrak{p}$.) We conclude using Theorem 2.

4. The problem and its solution for plane curves

In this section, we answer for plane curves to the following question:

How can we interpret the reducedness of $\mathcal{L}(X)$ in terms of the geometry of X?

Let us begin by a general result. Let k be a field of characteristic zero. Let $D \in \mathrm{Der}_k(A)$ and $f \in A$. One can consider the following power series

$$\varphi_D(f,T) := \sum_{i\in\mathbf{N}} \frac{D^i(f)}{i!} T^i, \tag{2}$$

with the usual conventions $0! = 1! = 1$ and $D^0 = \mathrm{Id}_A$. Let us set $\varphi_D : A \to A[\![T]\!]$ to be the map defined by $f \mapsto \varphi_D(f,T)$. It is an easy exercise to verify that, by Leibniz rule, the map φ_D is a morphism of k-algebras.

The result below links, in a direct manner, the arcs scheme of X and the derivation module $\mathrm{Der}_k(A)$.

Lemma 4.1. *Let k be a field of characteristic zero. Let $X = \mathrm{Spec}(A)$ be a variety over k. The map*

$$\varphi : \mathrm{Der}_k(A) \to \mathrm{Hom}_k\big(X, \mathcal{L}(X)\big)$$

defined by $D \mapsto \varphi_D$ is an injection.

Proof. Since X is affine, we have a natural bijection:

$$\mathrm{Hom}_k\big(X, \mathcal{L}(X)\big) \cong \mathrm{Hom}_k\big(\mathrm{Spec}\big(A[\![T]\!]\big), \mathrm{Spec}(A)\big) \cong \mathrm{Hom}_k\big(A, A[\![T]\!]\big).$$

So the map φ is well defined.

Let us prove that φ is injective. Let D_1, D_2 be two derivations over k of A such that $\varphi_{D_1} = \varphi_{D_2}$. We have to prove that $D_1 = D_2$. Let $g \in A$. We have to prove that $D_1(g) = D_2(g)$. As $D_i(g)$ is the first coefficient of $\varphi_{D_i}(g,T)$, for $i \in \{1,2\}$, we conclude. □

Example 3. Let A be a k-algebra of finite type and $X = \mathrm{Spec}(A)$. The trivial derivation D of A defined by $D(a) = 0$, for any $a \in A$, corresponds to the trivial morphism of k-algebras $A \to A[\![T]\!]$ which sends $a \in A$ to $a \in A[\![T]\!]$. Or, equivalently, it also corresponds to the morphism $s_0 : X \to \mathcal{L}(X)$, which is the section of the projection $\pi_0 : \mathcal{L}(X) \to X$.

Now, let x be a rational point of X. Then the *evaluation morphism* $\mathrm{ev}_x : A \to k$, defined by $f \mapsto f(x)$, induces a morphism of k-algebras

$$\mathrm{ev}_x : A[\![T]\!] \to k[\![T]\!]$$

that sends $\sum_{i\geqslant 0} f_i T^i$ to $\sum_{i\geqslant 0} \mathrm{ev}_x(f_i) T^i$. Then, composing this morphism of k-algebras by φ_D, we can associate with a derivation D over k of A a family of arcs on X, defined by $(\mathrm{ev}_x \circ \varphi_D)_{x\in X(k)}$.

4.2. Preliminary results

Let k be a field of characteristic zero. Let X be a variety over k. Some parts of Question 1 are easy.

Lemma 4.3. *Let k be a field of characteristic zero. Let X be a variety over k. If $\mathcal{L}(X)$ is reduced, then is X reduced.*

Proof. Assume that X is not reduced. By considering a well-chosen open subscheme U of X, we can assume that X is affine (and still non-reduced). Then, by considering the projection $\pi_0 : \mathcal{L}(X) \to X$ and its section $s_0 : X \to \mathcal{L}(X)$, we obtain a composition of morphisms of k-algebras between global sections rings

$$A \to A_\infty \to A$$

that is equal to Id_A. Let $f \in A$ be a non-trivial nilpotent element. Then the first morphism sends it to zero in A_∞, by assumption. By composition, we obtain $\mathrm{Id}_A(f) = 0$, with $f \neq 0$. That is a contradiction. □

On the other hand, it is not difficult to find examples of reduced varieties whose associated arcs scheme is non-reduced.

Example 4. Let k be a field of characteristic zero. Let $X = \mathrm{Spec}(A)$ be the hypersurface of $\mathrm{Spec}(k[x, y])$ defined by the datum of the polynomial xy. The associated arcs scheme $\mathcal{L}(X)$ is not reduced. Indeed, the element $x_0 y_1$ is a non-trivial nilpotent element of A_∞. We will justify this point in Sections 4.6, 4.8.

Example 5. Let $X = \mathrm{Spec}(A)$ be the hypersurface of $\mathrm{Spec}(k[x, y])$ defined by the datum of the polynomial $x^3 - y^2$. The associated arcs scheme $\mathcal{L}(X)$ is not reduced. Indeed, the element $3y_0x_1 - 2x_0y_1$ is a non-trivial nilpotent element of A_∞. We will justify this point in Sections 4.6, 4.8.

Remark 2. We will show in Section 4.8 that these examples are two illustrations of our method that allows to exhibit nilpotent elements of A_∞.

To conclude this discussion, we see that the reducedness of X is not the "good" property that characterizes that of $\mathcal{L}(X)$. This last property seems to be linked to the singularities of X.

Remark 3. The main result in [4] shows, in particular, that locally complete intersection varieties with rational singularities have reduced arc scheme.

Now, we establish some technical useful results.

Proposition 4.4. *Let k be a field of characteristic zero. Let $F \in k[x, y]$ be a non-constant polynomial and let $D \in \mathrm{Der}_k(k[x, y])$ be a non-trivial derivation such that $D(F) \in (F)$ and $D = a\partial_x + b\partial_y$, with $a, b \in k[x, y]$. Let X be the hypersurface of $\mathrm{Spec}(k[x, y])$ defined by the polynomial F. Let K be a field extension of k and let $\alpha \in \mathcal{L}(X)(K)$ be an arc, represented by $(\gamma(T), \sigma(T)) \in K[\![T]\!]^2$. Suppose that*

$$\partial_x F\big(\gamma(T), \sigma(T)\big) \neq 0 \quad \textit{or} \quad \partial_y F\big(\gamma(T), \sigma(T)\big) \neq 0$$

in $K[\![T]\!]$. Then we have

$$\begin{vmatrix} a(\gamma(T), \sigma(T)) & \gamma'(T) \\ b(\gamma(T), \sigma(T)) & \sigma'(T) \end{vmatrix} = 0.$$

Proof. By definition of $\mathcal{L}(X)$, we have $F(\gamma(T), \sigma(T)) = 0$. By derivating with respect to T this relation, we obtain

$$\gamma'(T)\partial_x F\big(\gamma(T), \sigma(T)\big) + \sigma'(T)\partial_y F\big(\gamma(T), \sigma(T)\big) = 0. \tag{3}$$

Since $D(F) \in (F)$, we also obtain

$$a\big(\gamma(T), \sigma(T)\big)\partial_x F\big(\gamma(T), \sigma(T)\big) + b\big(\gamma(T), \sigma(T)\big)\partial_y F\big(\gamma(T), \sigma(T)\big) = 0.$$

This relation added with (3) implies by a basic linear algebra argument that

$$\begin{vmatrix} a(\gamma(T), \sigma(T)) & \gamma'(T) \\ b(\gamma(T), \sigma(T)) & \sigma'(T) \end{vmatrix} = 0, \tag{4}$$

since $\partial_x F(\gamma(T), \sigma(T))$ and $\partial_x F(\gamma(T), \sigma(T))$ are not both zero. □

Remark 4. By [2, Chap. 0, §15 Proposition 10, §16 Corollary 3], note that, for any point $x \in X$, there exists a non-constant arc $(\gamma(T), \sigma(T)) \in (k^{\mathrm{alg}}[\![T]\!])^2$ on X such that $(\gamma(0), \sigma(0))$ corresponds to x. For a smooth point $x \in X$, the fiber $\pi_0^{-1}(x)$ can be identified with the affine space $\mathrm{Spec}(k[(Z_i)_{i \in \mathbf{N}}])$. If $k \subset \mathbf{C}$, and if $F \in k[x, y]$ is supposed to be analytically irreducible, i.e., irreducible in $\mathbf{C}\{x, y\}$, we can also conclude by the existence of Puiseux expansions.

When $k \subset \mathbf{C}$, it is not difficult to interpret the statement of Proposition 4.4 as follows. Let $\omega \in \Omega^1_{k[x,y]/k}$ be the Kähler differential $\omega = b\,dx - a\,dy$. Let $\alpha := (\gamma(T), \sigma(T)) \in \mathbf{C}[\![T]\!]^2$ be an arc such that $\partial_x F(\gamma(T), \sigma(T)) \neq 0$ or $\partial_y F(\gamma(T), \sigma(T)) \neq 0$. Thus

$$\alpha^* \omega = 0$$

in $\widehat{\Omega}^1_{\mathbf{C}[\![t]\!]/\mathbf{C}}$. That exactly means that $\alpha(T)$ is a (formal) *integral curve* of ω, passing through $\alpha(0)$.

Proposition 4.5. *Let k be a subfield of $\mathbf{C}$. Let $F \in k[x, y]$ be a non-constant reduced polynomial and let $D \in \mathrm{Der}_k(k[x, y])$ be a non-trivial derivation such that $D(F) \in (F)$ and $D = a\partial_x + b\partial_y$, with $a, b \in k[x, y]$. Let X be the hypersurface of $\mathrm{Spec}(k[x, y])$ defined by the polynomial F. Let $P \in X(\mathbf{C})$ be a singular point of X. If $(\alpha_0, \beta_0) \in \mathbf{C}^2$ corresponds to P, then*

$$\begin{cases} a(\alpha_0, \beta_0) = 0, \\ b(\alpha_0, \beta_0) = 0. \end{cases}$$

Proof. From the assumption, we deduce that $F \in \mathbf{C}[x, y]$ is reduced. Thus, we can assume that $k = \mathbf{C}$.

Let $\varphi = (\alpha(T), \beta(T)) \in \mathcal{L}(X)(\mathbf{C})$ be an arc of X. If $\partial_x F(\alpha(T), \beta(T)) = 0$ and $\partial_y F(\alpha(T), \beta(T)) = 0$, then $\alpha \in \mathcal{L}(\mathrm{Sing}(X))$, where $\mathrm{Sing}(X)$ denotes the non-smooth locus of X. Since F is reduced, the scheme $\mathcal{L}(\mathrm{Sing}(X))$ is isomorphic to a finite sum of points. In particular, φ is constant. Then, using Proposition 4.4, we deduce that any *non-constant* arc $\alpha(T)$ in $\mathcal{L}(X)(\mathbf{C})$ verifies $\alpha^*\omega = 0$, with $\omega = b\,dx - a\,dy$. If we consider $X(\mathbf{C})$ as an analytic variety embedded in $\mathbf{C}^2$, on which there acts a vector field D of $\mathbf{C}^2$, the above remark holds true in particular for the parameterizations of the analytic branches of F, in a neighborhood of (α_0, β_0) in $\mathbf{C}^2$ (for the usual topology of $\mathbf{C}^2$), as indicated in Remark 4.

Assume that $a(\alpha_0, \beta_0)$ and $b(\alpha_0, \beta_0)$ are not both equal to zero, i.e., D is smooth at (α_0, β_0). We deduce, by applying Cauchy–Lipschitz's theorem and Rectification Theorem for vector fields (see [1, Theorem 1.1, Theorem 1.18]) to ω (or D) that (α_0, β_0) cannot be a singular point in $X(\mathbf{C})$. That is a contradiction. □

4.6. How to find nilpotent elements on affine arcs spaces

Let k be a field of characteristic zero. Let $X = \mathrm{Spec}(A)$ be a variety over k. The aim of this paragraph is to introduce a general method that allows to exhibit (non-trivial) nilpotent elements of A_∞. This method is new and related to the structure of the derivation module of A.

Theorem 3. *Let k be a subfield of $\mathbf{C}$. Let $F \in k[x, y]$ be a non-constant reduced polynomial and let $D \in \mathrm{Der}_k(k[x, y])$ be a non-trivial derivation such that $D(F) \in (F)$ and $D = a\partial_x + b\partial_y$, with $a, b \in k[x, y]$. Let X be the hypersurface of $\mathrm{Spec}(k[x, y])$ defined by the polynomial F.*

- a) *The element $(ay_1 - bx_1) \in k[x, y]_\infty$ belongs to $\{F\}$.*
- b) *If there exists no element $g \in A$ such that the induced derivation $\overline{D} = g\delta_F$ in $\mathrm{Der}_k(A)$, then the element $(ay_1 - bx_1) \in k[x, y]_\infty$ does not belong to $[F]$. Equivalently, the element $(ay_1 - bx_1) \in A_\infty$ is a non-trivial nilpotent element of A_∞.*

Proof. Let us set some notations. In our situation, we have

$$\begin{cases} \mathcal{L}(X) = \operatorname{Spec}\left(\dfrac{k[x_0, y_0, x_1, y_1, \ldots, x_n, y_n, \ldots]}{[F]}\right), \\ [F] = \big(F, \Delta(F), \Delta^2(F), \ldots\big), \\ \{F\} = \sqrt{[F]}. \end{cases}$$

a) We prove the following stronger result

$$(ay_1 - bx_1) \in \sqrt{\big(F, \Delta(F)\big)} \cdot k[x_0, y_0, x_1, y_1].$$

By Hilbert's Nullstellensatz, it is equivalent to prove that, for any $(\alpha_0, \beta_0, \alpha_1, \beta_1) \in \mathbf{C}^4$ such that $F(\alpha_0, \beta_0) = 0$ and $\Delta(F)(\alpha_0, \beta_0, \alpha_1, \beta_1) = 0$, we have

$$\alpha_1 b(\alpha_0, \beta_0) - \beta_1 a(\alpha_0, \beta_0) = \begin{vmatrix} b(\alpha_0, \beta_0) & \beta_1 \\ a(\alpha_0, \beta_0) & \alpha_1 \end{vmatrix} = 0. \tag{5}$$

Let $(\alpha_0, \beta_0, \alpha_1, \beta_1) \in \mathbf{C}^4$ be such an element. Two cases occur.

Assume first that (α_0, β_0) is a singular point of X. In this case, we conclude by Proposition 4.5.

Assume now that (α_0, β_0) is a smooth point of X. Let K be an extension of $\mathbf{C}$ and $\varphi := (\alpha(T), \beta(T)) \in (K[\![T]\!])^2$ be an arc on X with base point (α_0, β_0). Then $\partial_x F(\varphi) \in K[\![T]\!]$, with value in $T = 0$ equal to $\partial_x F(\alpha_0, \beta_0)$, by Taylor expansion. Then any arc $(\alpha(T), \beta(T)) \in (K[\![T]\!])^2$ on X with base point (α_0, β_0) verifies that $\partial_x F(\alpha(T), \beta(T))$ and $\partial_x F(\alpha(T), \beta(T))$ are not both zero. By [12, Lemme 3.4.2] and by the previous remark, we conclude that we can find an arc $(\alpha(T), \beta(T)) \in K[\![T]\!]^2$ such that $F(\alpha(T), \beta(T)) = 0$, and $(\alpha(0), \beta(0)) = (\alpha_0, \beta_0)$, $(\alpha'(0), \beta'(0)) = (\alpha_1, \beta_1)$, and such that $\partial_x F(\alpha(T), \beta(T))$ and $\partial_x F(\alpha(T), \beta(T))$ are not both zero, for some field extension K of $\mathbf{C}$. In particular, by applying Proposition 4.4 to the arc $(\alpha(T), \beta(T))$, we conclude that

$$\beta'(T) \cdot a\big(\alpha(T), \beta(T)\big) - \alpha'(T) \cdot b\big(\alpha(T), \beta(T)\big) = 0. \tag{6}$$

By specializing T to 0 in (6), we prove (5).

b) Assume that $ay_1 - bx_1 \in [F] \cdot k[x, y]_\infty$. It means that there exists an integer s, $s \geqslant 1$, $Q_0, Q_1, \ldots, Q_s \in k[x, y]_\infty$ such that

$$a(x_0, y_0)y_1 - b(x_0, y_0)x_1 = Q_0 \cdot F(x_0, y_0) + Q_1 \cdot \Delta(F) + \cdots + Q_s \cdot \Delta^s(F).$$

By the definition of the differential ring $k[x, y]_\infty$, there exists an integer r, $r \geqslant s$, such that we can consider this equality in the ring of polynomials $k[x_0, y_0, \ldots, x_r, y_r]$. We interpret below such an equation.

Firstly, note that the right hand term is a sum of polynomials that cannot contain a polynomial $R \in k[x_0, y_0]$. Indeed, to see that fact, one can specialize this equality by putting $x_1 = y_1 = \cdots = x_r = y_r = 0$.

Secondly, note that the polynomials of the form $\Delta^i(F)$ (and so $Q_i \Delta^i(F)$), for $i \geqslant 2$, are sums of monomials that are either divisible by x_ℓ or y_ℓ, with $\ell \geqslant 2$, or divisible by $x_1 y_1$ or x_1^ℓ or y_1^ℓ, with $\ell \geqslant 2$.

Let us set $P \in k[x_0, y_0, \ldots, x_r, y_r]$ to be the sum of the monomials of the polynomial

$$Q_0 \cdot F + Q_1 \cdot \Delta(F) + \cdots + Q_s \cdot \Delta^s(F)$$

that are either divisible by x_ℓ or y_ℓ, with $\ell \geqslant 2$, or divisible by $x_1 y_1$ or x_1^ℓ or y_1^ℓ, with $\ell \geqslant 2$. In particular, all the monomials of $Q_i \Delta^i(F)$, for any i, $i \geqslant 2$, appear in the expression of P. Then we can write $a(x_0, y_0)y_1 - b(x_0, y_0)x_1$ as

$$(x_1 q_{1,0} + y_1 q_{0,1}) \cdot F + q_1 \cdot \big(\partial_x(F)x_1 + \partial_y(F)y_1\big) + P \tag{7}$$

with $q_{1,0}, q_{0,1}, q_1 \in k[x_0, y_0]$. Since the monic monomials form a basis of the k-vector space $k[x_0, y_0, \dots, x_r, y_r]$, we deduce from (7) that $P = 0$ and

$$\begin{cases} a(x_0, y_0) = q_1(x_0, y_0)\partial_y F(x_0, y_0) + q_{0,1}(x_0, y_0)F(x_0, y_0), \\ b(x_0, y_0) = -\big(q_{1,0}(x_0, y_0)F(x_0, y_0) + q_1(x_0, y_0)\partial_x F(x_0, y_0)\big). \end{cases} \tag{8}$$

It follows from (8) that:

$$D = a\partial_x + b\partial_y = q_1 \cdot (\partial_y F \partial_x - \partial_x F \partial_y) + F \cdot (q_{0,1}\partial_x - q_{1,0}\partial_y).$$

So D induces a derivation $\overline{D}$ over k of A such that $\overline{D} = \bar{q}_1 \delta_F$ in $\mathrm{Der}_k(A)$. That is a contradiction. □

Remark 5. Note that, in particular, Theorem 3 exhibits a (non-trivial) nilpotent element of $\mathcal{L}_1(X)$. But, the very specific and simple form of that element guarantees, as we have shown in the proof of Theorem 3b), that it is not trivial in A_∞.

Remark 6. It follows from [4, Proposition 4.12] that if X is a singular, locally complete intersection curve, then $\mathcal{L}_1(X)$ is non-reduced.

4.7. The main theorem

We are now able to prove the following theorem, that is a direct consequence of the study of Section 4.6 and Theorem 3.

Theorem 4. *Let k be a subfield of* $\mathbf{C}$. *Let X be a plane curve over k. Then X is smooth over k if and only if the k-scheme $\mathcal{L}(X)$ is reduced.*

Proof. If X is smooth over k, it follows from Proposition 3.2 that $\mathcal{L}(X)$ is reduced.

Conversely, assume that X has singular points. Two cases occur. First assume that X is non-reduced. Then $\mathcal{L}(X)$ is non-reduced, by Lemma 4.3. Now assume that X is a reduced, singular, plane curve over k. It is sufficient to find an affine open subscheme U of X such that $\mathcal{O}_{\mathcal{L}(U)}(\mathcal{L}(U))$ has a non-trivial nilpotent element, by Proposition 3.2b). Let x be a singular point of X. Let U_i, for $i \in \{0, 1, 2\}$, be the standard charts of $\mathbf{P}^2_k$. Assume that $U := U_0$ contains x. On U, the curve X is a reduced, singular, affine plane curve over k, say $\mathrm{Spec}(k[x, y]/(F))$. Let us set $A = k[x, y]/(F)$. Let $\overline{D} \in \mathrm{Der}_k(A)$ be a derivation over k of A such that, for any $g \in A$, $\overline{D} \neq g\delta_F$. Note that such a derivation exists, by Lipman's theorem (see Theorem 2 and Remark 1). Let $D \in \mathrm{Der}_k(k[x, y])$ be a lifting of $\overline{D}$. It is non-trivial, since $\overline{D}$ is non-trivial. We conclude, using Theorem 3, that A_∞ has a non-trivial nilpotent element. □

4.8. Examples

Let k be a subfield of $\mathbf{C}$. Let $X = \mathrm{Spec}(A)$ be a reduced affine plane curve over k defined by the datum of a non-constant reduced polynomial $F \in k[x, y]$. Following Theorem 3, a natural question appears firstly.

How to find a derivation D over k of A which is not multiple of δ_F?

This kind of objects exists theoretically by Lipman's Theorem 2, but is a priori, and in general, not easy to compute explicitly. In [11],[1] we have shown, in this direction, the following statement.

Proposition 4.9. *Let $F \in \mathbf{C}[x, y]$ be a polynomial whose degree in x is at least 3. Let us set $R_F := \mathrm{Result}_x(F, \partial_x F)$. Assume that $R_F \neq 0$. There exists a unique couple of polynomials (U, V) respectively of degree (in x) at most $d-2$ and $d-1$ such that*

$$xR_F\partial_y F = UF + V\partial_x F.$$

The derivation $D := xR_F\partial_y - V\partial_y$ is called the resultant derivation *(in x). The derivation obtained by dividing xR_F and V by their common factors is the* reduced resultant derivation*.*

Now, we illustrate Theorem 3 by examples. In Examples 6 and 8, the derivations inducing "candidates" for nilpotent elements are exhibited by Proposition 4.9.

Example 6. Let $X = \mathrm{Spec}(A)$ be the hypersurface of $\mathrm{Spec}(\mathbf{C}[x, y])$ defined by the datum of the polynomial $F = x^3 - y^2 - 1$. The curve X is smooth over k. The (reduced) resultant derivation is $D := 3(1 + y^2)\partial_y + 2xy\partial_x$. We see in particular that $\overline{D} = -x\delta_F$ in $\mathrm{Der}_k(A)$. Here, Theorem 3 only produces *trivial* nilpotent elements, since $\mathcal{L}(X)$ is reduced, because of the smoothness of X.

Example 7. Let $X = \mathrm{Spec}(A)$ be the hypersurface of $\mathrm{Spec}(\mathbf{C}[x, y])$ defined by the datum of the polynomial $F = xy$. The curve X is reduced and the point $(0, 0)$ is the unique singular point. We verify that x_0y_1 is a nilpotent element of A_∞. Indeed,

$$x_0y_1 \cdot \Delta(x_0y_0) = x_0y_1(x_0y_1 + x_1y_0) = (x_0y_1)^2 + x_1y_1x_0y_0.$$

It implies that $x_0y_1 \in \{F\}$. It remains to show that $x_0y_1 \notin [F]$. Assume that there exist $Q_0, Q_1, \ldots, Q_s$ such that

$$x_0y_1 = Q_0F + Q_1\Delta(F) + \cdots + Q_s\Delta^s(F).$$

An easy observation forces that, in fact,

$$x_0y_1 = p_0x_1F + q_0y_1F + (x_0y_1 + x_1y_0)r_0, \tag{9}$$

with $p_0, q_0, r_0 \in \mathbf{C}[x_0, y_0]$. It implies that x_0 must divide r_0 and that $r_0 + q_0y_0 = 1$. This is impossible.

Now we justify the assertion by our method. Let $D := x\partial_x$. With this choice of derivation, we see directly by Theorem 3 that $x_0y_1 \in \{F\}$. Assume that there exists $g \in A$ such that $\overline{D} = g\delta_F$ in $\mathrm{Der}_k(A)$. In this case, $\overline{D}(y) = 0 = -yg$ in A. Thus $g \in xA$ and $\delta_F = x\tilde{g}\partial_x$ in $\mathrm{Der}_k(A)$. Thus $\overline{D}(x) = x = xg$ in A and $(1 - g) \in yA$. That is impossible.

Example 8. Let $X = \mathrm{Spec}(A)$ be the hypersurface of $\mathrm{Spec}(\mathbf{C}[x, y])$ defined by the datum of the polynomial $F = x^3 - y^2$. The curve X is reduced and the point $(0, 0)$ is the unique singular point. We verify that $3y_0x_1 - 2x_0y_1$ is a nilpotent element of A_∞. Division algorithm in $\mathbf{C}[x, y]$ gives the following formula for $(3y_0x_1 - 2x_0y_1)^3$

$$(3y_0x_1 - 2x_0y_1) \cdot \left(-9x_1^2F + 3x_1x_0\Delta(F)\right) - 8y_1^3F + 4y_0y_1^2\Delta(F).$$

[1] This construction generalizes that of [6,7], where $n = 3$. In the 3-dimensional case, see also [8–10] for an interpretation of that derivation in terms of web geometry.

So $3y_0x_1 + 2x_0y_1 \in \{F\}$. It remains to show that $3y_0x_1 - 2x_0y_1 \notin [F]$. We conclude by the same way as in the previous example. Indeed, once again we must have a system of form:

$$3y_0x_1 - 2x_0y_1 = p_0x_1F + q_0y_1F + (x_0y_1 + x_1y_0)r_0, \tag{10}$$

with $p_0, q_0, r_0 \in \mathbf{C}[x_0, y_0]$. This is impossible, because of the degrees in x_0 and y_0 in F.

Now we justify the assertion by our method. Here, the (reduced) resultant derivation is $D := 3y\partial_y + 2x\partial_x$. With this choice of derivation, we see directly by Theorem 3 that $3y_0x_1 - 2x_0y_1 \in \{F\}$. Assume that there exists $g \in A$ such that $\overline{D} = g\delta_F$ in $\mathrm{Der}_k(A)$. Let $\overline{A}$ be the integral closure of A. By the general properties of derivations, we know that $\overline{D}$ extends in a derivation over $\mathbf{C}$ of $\overline{A}$. The same is true for δ_F and for the relation $\overline{D} = g\delta_F$. We can compute directly these liftings. We know that $\overline{A} = k[y/x]$. Let us set t for y/x in $\overline{A}$. It follows that $y = t^3$, $x = t^2$, $\overline{D} = t\partial_t$ and $\delta_F = \partial_t$. But the regular function t of $\overline{A}$ does not come from a regular function of A. A contradiction.

Acknowledgment

I would like to thank the referee for his careful reading of the article, and for pointing out Remarks 3 and 6.

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